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The top cohomology class of certain spaces

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Abstract

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We give an explicit formula for a cycle representing a basis for the cohomology class of highest degree of certain spaces, including the compact homogeneous spaces.

Introduction

A topological space S is *rationally elliptic* [2] if the spaces $H^*(S;\mathbb{Q})$ and $\pi_*(S)\otimes\mathbb{Q}$ are both finite-dimensional. Compact homogeneous spaces are classical examples of such spaces.

It is known [5, Theorem 3] that the (rational) cohomology of a 1-connected elliptic space S is a *Poincaré duality algebra*. One can compute the degree of its *top class* (a top class of a Poincaré duality algebra $H=\sum_{i=0}^N H^i$ is a generator of H^N) by the formula:

$$\sum_{k\text{ odd}} k \dim \pi_k(S)\otimes\mathbb{Q} - \sum_{k\text{ even}} (k-1)\dim \pi_k(S)\otimes\mathbb{Q}\,.$$

In this paper we give an explicit formula for a cycle representing the top class of certain elliptic spaces, including the homogeneous spaces.

For that, we shall rely on the connection between Sullivan's theory of minimal models and rational homotopy theory for which [2], [6] and [10] are standard

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references. Here we recall some notation and conventions:

We shall work over a field $\mathbb{K}$ of characteristic zero unless stated otherwise. A *KS-complex* is a commutative differential graded algebra (CDGA) $(\Lambda X, d)$ where

$$\Lambda X = \mathrm{Exterior}(X^{\mathrm{odd}}) \otimes \mathrm{Symmetric}(X^{\mathrm{even}})$$

is the free commutative algebra generated by the graded vector space X which has a well ordered basis $\{x_\alpha\}$ such that $dx_\alpha \in \Lambda X_{<\alpha}$. If $\deg x_\alpha < \deg x_\beta$ implies $\alpha < \beta$, the KS-complex $(\Lambda X, d)$ is *minimal*. When $(\Lambda X, d)$ is 1-connected $(X^0 = X^1 = 0)$, this is equivalent to saying that $dX \subset \Lambda^{\geq 2} X$. Given the CDGA $A(S)$ of differential forms on a topological space S [10], there exists a minimal KS-complex $(\Lambda X, d)$ and a quism (CDGA morphism inducing homology isomorphism) $\varphi : (\Lambda X, d) \xrightarrow{\simeq} A(S)$. This is the *minimal model* of S and is unique up to isomorphism [6, Chapter 6]. Since $A(S)$ and $C^*(S; \mathbb{K})$ are connected by a chain of quisms, $H^*(\Lambda X, d)$ and $H^*(S; \mathbb{K})$ are naturally identified.

Definition. A KS-complex $(\Lambda Z, d)$ is said to be a *pure tower* if $dZ^{\mathrm{even}} = 0$ and $dZ^{\mathrm{odd}} \subset \Lambda(Z^{\mathrm{even}})$. If $\dim Z < \infty$, $(\Lambda Z, d)$ is a *finite pure tower.*

Remark. Homogeneous spaces are examples of spaces whose minimal models are pure towers [4, Chapter XI] or [1].

Given a pure tower $(\Lambda Z, d)$ we shall denote $X = Z^{\mathrm{even}}$ and $Y = Z^{\mathrm{odd}}$. Let $(\Lambda Z, d) = (\Lambda(x_1, \ldots, x_n) \otimes \Lambda(y_1, \ldots, y_m), d)$ be a finite pure tower and let $f_i = dy_i$, $i = 1, \ldots, m$. In [5, Section 3] it is shown that $H^*(\Lambda Z, d)$ is finite-dimensional if and only if the algebra

$$\mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_m),$$

where $(f_1, \ldots, f_m)$ denotes the ideal generated by the polynomials $f_1, \ldots, f_m$, is finite-dimensional. Then it is clear that $m \geq n$.

Assume $\dim H^*(\Lambda Z, d) < \infty$ and write

$$f_i = a_i^1 x_1 + a_i^2 x_2 + \cdots + a_i^{n-1} x_{n-1} + a_i^n x_n\,, \quad i = 1, \ldots, m\,, \tag{1}$$

where a_i^j are polynomials in the variables $x_j, x_{j+1}, \ldots, x_n$. Consider the matrix

$$A = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n-1} & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^{n-1} & a_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_m^1 & a_m^2 & \cdots & a_m^{n-1} & a_m^n \end{bmatrix}$$

and let $P \in \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the polynomial defined as follows:

$$P = \sum_{1 \le i_1 < \cdots < i_n \le m} (-1)^{i_1 + \cdots + i_n} P_{i_1 \ldots i_n} y_1 \cdots \hat{y}_{i_1} \cdots \hat{y}_{i_n} \cdots y_m ,$$

in which $P_{i_1 \ldots i_n}$ is the determinant of the matrix of order n formed by the rows $i_1, \ldots, i_n$ of A,

$$P_{i_1 \ldots i_n} = \begin{vmatrix} a^1_{i_1} & \ldots & a^n_{i_1} \\ \vdots & \ddots & \vdots \\ a^1_{i_n} & \ldots & a^n_{i_n} \end{vmatrix} .$$

Then, we prove the following:

Theorem. *P is a cycle representing the top cohomology class of $H(\Lambda Z, d)$.*

Examples. (1) Consider the homogeneous space $U(4)/(U(2) \times U(2))$. Its minimal model is a finite pure tower of the form [4, Chapter XI.4]

$$\begin{aligned} &(\Lambda(x_2, x_4, y_5, y_7), d) , \\ &dx_2 = dx_4 = 0 , \quad dy_5 = x_2^3 - 2x_2x_4 , \quad dy_7 = x_4^2 - x_2^2 x_4 , \end{aligned}$$

in which subscripts denote degrees. Then, we have

$$A = \begin{bmatrix} x_2^2 - 2x_4 & 0 \\ -x_2x_4 & x_4 \end{bmatrix}$$

and by the theorem above we can compute a generator of the top cohomology class:

$$[x_2^2 x_4 - 2x_4^2] \in H^8(U(4)/(U(2) \times U(2))) .$$

(2) Consider now the space $\mathrm{SU}(6)/(\mathrm{SU}(3) \times \mathrm{SU}(3))$ whose minimal model is

$$\begin{aligned} &(\Lambda(x_4, x_6, y_7, y_9, y_{11}), d) , \\ &dy_7 = -x_4^2 , \quad dy_9 = -2x_4x_6 , \quad dy_{11} = -x_6^2 . \end{aligned}$$

In this case:

$$A = \begin{bmatrix} -x_4 & 0 \\ -2x_6 & 0 \\ 0 & -x_6 \end{bmatrix} .$$

Then, $H^{19}(\mathrm{SU}(6)/(\mathrm{SU}(3) \times \mathrm{SU}(3)))$ is generated by $[x_4x_6y_9 - 2x_6^2y_7]$.

1. Proof of the Theorem

The rest of the paper is devoted to the proof of the theorem stated in the Introduction. For that we shall use some concepts and results from differential homological algebra for which we refer to [3, Appendix] or [8, Section 1]. Here we simply recall basic definitions:

Let A be a DGA (Differential Graded Algebra). An A-module M is A-semifree if there exists a filtration of A-submodules $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots$ such that $M = \bigcup_i F_i$ and for each i, F_i/F_{i-1} is A-free on a basis of cycles. A quism of A-modules $P \xrightarrow{\simeq} M$ is called a *semifree resolution* of M if P is semifree. Let M,N be A-modules and $P \xrightarrow{\simeq} M$ a semifree resolution. Define $\mathrm{Ext}_A(M, N) = H(\mathrm{Hom}_A(P, N))$.

Beginning with the proof of our Theorem, consider the KS-complex $(\Lambda(x_1, \ldots, x_n), 0)$. By [3, Section 3], $\mathrm{Ext}_{\Lambda(x_1,\ldots,x_n)}(\mathbb{Q}, \Lambda(x_1, \ldots, x_n))$ is a 1-dimensional $\mathbb{K}$-vector space and a basis element is represented by the homomorphism of $\Lambda(x_1, \ldots, x_n)$-modules,

$$g : (\Lambda(x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n), d) \to (\Lambda(x_1, \ldots, x_n), 0) ,$$
$$g(1 \otimes \Phi) = 0 , \quad \Phi \in \Lambda^{<n}(\bar{x}_1, \ldots, \bar{x}) , \quad g(\bar{x}_1 \cdot \bar{x}_2 \cdots \bar{x}_n) = 1 ,$$

in which $(\Lambda(x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n), d)$, $d\bar{x}_i = x_i$, is a model of the projection

$$(\Lambda(x_1, \ldots, x_n), 0) \to (\mathbb{K}, 0)$$

and thus it is a $\Lambda(x_1, \ldots, x_n)$-semifree solution of $\mathbb{K}$.

On the other hand, the projection

$$(\Lambda(x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n) \otimes \Lambda(y_1, \ldots, y_m), d)$$
$$\xrightarrow{\simeq} (\Lambda(y_1, \ldots, y_m), 0)$$

is a quism, and therefore, there exists an element $\Phi_0 \in \Lambda^+(X \oplus \bar{X}) \otimes \Lambda Y$ such that $d\Phi_0 = d(y_1 \cdot y_2 \cdots y_m)$. Write

$$\Phi_0 = \sum_k \gamma_k \beta_k , \qquad \gamma_k \in \Lambda^+(X \oplus \bar{X}) , \quad \beta_k \in \Lambda Y .$$

Lemma 1.1. *A basis element of* $\mathrm{Ext}_{\Lambda Z}(\mathbb{Q}, \Lambda Z) \cong \mathbb{K}$ *is represented by a homomorphism of* ΛZ*-modules*

$$h : (\Lambda X \otimes \Lambda Y \otimes \Lambda \bar{X} \otimes \Lambda \bar{Y}, d) \to (\Lambda X \otimes \Lambda Y, d)$$

satisfying $h(1) = -\sum_k g(\gamma_k)\beta_k$.

Proof. First observe that a basis element of $\mathrm{Ext}_{\Lambda(y_1,\dots,y_m)}(\mathbb{Q}, \Lambda(y_1,\dots,y_m))$ is represented by $f : \Lambda(y_1,\dots,y_m,\bar{y}_1,\dots,\bar{y}_m)\to \Lambda(y_1,\dots,y_m)$,

$$f(1) = y_1\cdot y_2\cdots y_m\,,\qquad f(\Psi)=0\,,\quad \Psi\in \Lambda Y\otimes \Lambda^+\bar{Y}\,.$$

Now, since $H^*(\Lambda(y_1,\dots,y_m),0)$ is finite-dimensional, we can apply [8, Theorem A] to get an isomorphism

$$\varphi : \mathrm{Ext}_{\Lambda X}(\mathbb{Q},\Lambda X)\otimes \mathrm{Ext}_{\Lambda Y}(\mathbb{Q},\Lambda Y)\xrightarrow{\cong}\mathrm{Ext}_{\Lambda Z}(\mathbb{Q},\Lambda Z)\,.$$

In the proof of [9, Theorem 3.3] there is an explicit formula for φ in this particular setting:

$$\varphi([g]\otimes[f]) = [\tilde{g}\circ\tilde{f}]\,,$$

in which $\tilde{g} : (\Lambda X\otimes\Lambda\bar{X}\otimes\Lambda Y, d)\to(\Lambda X\otimes\Lambda Y, d)$ is defined by $\tilde{g} = g\otimes 1_{\Lambda Y}$ and

$$\tilde{f} : (\Lambda X\otimes\Lambda\bar{X}\otimes\Lambda Y\otimes\bar{Y}, d)\to(\Lambda X\otimes\Lambda\bar{X}\otimes\Lambda Y, d)$$

is a $(\Lambda X\otimes\Lambda\bar{X}\otimes\Lambda Y, d)$-linear map that makes the diagram

$$\begin{array}{ccc}
\Lambda X\otimes\Lambda\bar{X}\otimes\Lambda Y & \xrightarrow{\simeq} & \Lambda Y\\
\tilde{f}\uparrow & & \uparrow f\\
\Lambda X\otimes\Lambda\bar{X}\otimes\Lambda Y\otimes\Lambda\bar{Y} & \xrightarrow{\simeq} & \Lambda Y\otimes\Lambda\bar{Y}
\end{array}$$

commutative. Hence $[\tilde{g}\circ\tilde{f}]$ is a basis of $\mathrm{Ext}_{\Lambda Z}(\mathbb{K};\Lambda Z)$.

To finish, define $h=\tilde{g}\circ\tilde{f}$ and observe that a possible choice of $\tilde{f}(1)$ is $\tilde{f}(1) = f(1) - \Phi_0 = y_1\cdot y_2\cdots y_n - \sum_k \gamma_k\beta_k$. Therefore, $h(1) = -\sum_k g(\gamma_k)\beta_k$. $\square$

We return now to the proof of the Theorem. Note that $dy_p = d\Phi_p$, $\Phi_p\in\Lambda X\otimes\Lambda\bar{X}$, $p=1,\dots,m$, where

$$\Phi_p = a_p^1\bar{x}_1+\cdots+a_p^{n-1}\bar{x}_{n-1}+a_p^n\bar{x}_n$$

and a_p^i are the polynomials of (1).

On the other hand, it is a straightforward computation to show that

$$d(y_1\dots y_m) = d\Big(\sum_{i=1}^m(-1)^{i-1}A_i\Big)\,,$$

where

$$\begin{aligned}A_j &= \sum_{1\le i_1<\cdots<i_j\le m} y_1\cdots y_{i_1-1}\Phi_{i_1}y_{i_1+1}\cdots y_{i_j-1}\Phi_{i_j}y_{i_j+1}\cdots y_m\\
&= (-1)^{1+\cdots+j}\sum_{1\le i_1<\cdots<i_j\le m}(-1)^{i_1+\cdots+i_j}\Phi_{i_1}\cdots\Phi_{i_j}y_1\cdots\hat{y}_{i_1}\cdots\hat{y}_{i_j}\cdots y_m\,.\end{aligned}$$

Then, by Lemma 1.1, $h(1)$ can be written as follows:

$$h(1) = -\sum_{j=1}^{m} g\Big(\pm \sum_{1\le i_1<\cdots<i_j\le m} (-1)^{i_1+\cdots+i_j} \Phi_{i_1}\cdots\Phi_{i_j}\Big)$$

$$y_1\cdots\hat{y}_{i_1}\cdots\hat{y}_{i_j}\cdots y_m .$$

But, since $g(\Phi)=0$ whenever $\Phi\in\Lambda^{<n}(\bar{x}_1,\ldots,\bar{x}_n)$ and since $g(\bar{x}_1\cdot\bar{x}_2\cdots\bar{x}_n)=1$, we have:

$$h(1) = \pm\Big(\sum_{1\le i_1<\cdots<i_n\le m} (-1)^{i_1+\cdots+i_n} g(\Phi_{i_1}\cdots\Phi_{i_n})\Big) y_1\cdots\hat{y}_{i_1}\cdots\hat{y}_{i_n}\cdots y_m .$$

It is then easy to check that in fact, this polynomial coincides with P (up to the sign). To finish observe that since $H^*(\Lambda Z, d)$ is a Poincaré duality algebra and h represents the unique nonzero class in $\mathrm{Ext}_{\Lambda Z}(\mathbb{K}, \Lambda Z)$, the class $[h(1)]$ must be the top cohomology class of $H^*(\Lambda Z\, d)$ ([3, Sections 1 and 3] or [9, Remark 2.1.(2)]).

Note. Observe that in (1), the polynomials a_i^j are not canonically determined so they can be chosen in the best way to make the computations easier.

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